MATH6031 Lecture 1

§ Quantization

$$\boxed{\text{Classical system}} \Longrightarrow \boxed{\text{Quantum system}}$$

More precisely:

- Symplectic manifold (M, ω) $\rightsquigarrow$ $\mathcal{H}$ Hilbert space

nondegenerate closed 2-form

- $\boxed{f \in C^\infty(M) \rightsquigarrow \mathcal{O}_f \in \mathcal{A}_\hbar = \{\text{operators on } \mathcal{H}\}}$
- Lagrangian submfd $L \subset M$ $\rightsquigarrow$ $\psi_L \in \mathcal{H}$
- $\phi \in \mathrm{Sympl}(M, \omega) \longrightarrow \Phi \in \mathrm{Aut}(\mathcal{A}_\hbar)$

satisfying some nice properties:

- $1 \longmapsto \mathcal{O}_1 = \mathrm{Id} \in \mathcal{A}_\hbar$
- $[\mathcal{O}_f, \mathcal{O}_g] = i\hbar\, \mathcal{O}_{\{f,g\}}$ (Dirac)

Here, $\{f, g\}$ is the Poisson bracket on $C^\infty(M)$ induced by the symplectic structure ω:

$$\{f, g\} = \omega(X_f, X_g)$$

where X_f is the Hamiltonian vector field associated to f:

$$\Gamma(T^*M) \xrightarrow[\cong]{\lrcorner\,\omega} \Gamma(TM)$$

$\because \omega$ is nondegenerate

$$df \longleftrightarrow X_f$$

Fact: The no-go thm by Gronewold and Van Hove says that such a quantization is impossible.

§ Deformation Quantization

Relax the condition $[\mathcal{O}_f, \mathcal{O}_g] = i\hbar \mathcal{O}_{\{f,g\}}$

to $[\mathcal{O}_f, \mathcal{O}_g] = i\hbar \mathcal{O}_{\{f,g\}}$ + higer order terms in $\hbar$

and also view $\mathcal{A}_\hbar$ as a noncommutative deformation of the commutative algebra $(C^\infty(M), \cdot)$ usual multiplication of functions

Rmk The advantage of this approach is that now we can consider not just sympl. mfds but also Poisson mfds i.e. a smooth mfd M equipped with a Poisson bracket $\{\cdot,\cdot\}$ on $C^\infty(M)$
this means a Lie bracket on $C^\infty(M)$ s.t. $\{f, gh\} = \{f,g\}h + \{f,h\}g$

Def (Bayen-Flato-Frønsdal-Lichnerowicz-Sternheimer 1978)
Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold.

A (differential) star-product or a deformation quantization on M is an associative product $*$ on $C^\infty(M)[[t]]$ (where $t = i\hbar$) written as

$$f * g = \sum_{k=0}^{\infty} C_k(f,g)\, t^k$$

s.t.
- $C_k(\cdot,\cdot)$ is a bidifferential operator (locality)
- $C_0(f,g) = fg$ $\quad \forall f,g \in C^\infty(M)$
- $C_1(f,g) - C_1(g,f) = \{f,g\}$

$$[f,g]_* = i\hbar\{f,g\} + O(\hbar^2)$$

- $f * 1 = 1 * f = f$ $\quad \forall f \in C^\infty(M)[[t]]$

First main question: Do these star-products exist?

1983 De Wilde-Lecomte }: existence for general sympl. mfds
1985 Fedosov (or regular Poisson mfds)

1997 Kontsevich : existence for general Poisson mfds

Thm (Kontsevich 1997) Every Poisson manifold $(M, \{\cdot,\cdot\})$ admits a deformation quantization

§ Lie Theory

k : field of characteristic 0 (e.g. $k = \mathbb{C}$)

Consider a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ over k
(i.e. $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is an antisymmetric, bilinear pairing which satisfies the Jacobi identity)

Associated to $\mathfrak{g}$ are the following:

- the tensor algebra
$$T^\bullet(\mathfrak{g}) = \bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n} \qquad (\text{here } \mathfrak{g}^{\otimes 0} = k)$$
(as a graded algebra)

- the symmetric algebra
$$S^\bullet(\mathfrak{g}) = T^\bullet(\mathfrak{g}) / \mathcal{I}$$
where $\mathcal{I}$ is generated by $x \otimes y - y \otimes x$ for $x, y \in \mathfrak{g}$

- the universal enveloping algebra
$$U^\bullet(\mathfrak{g}) = T^\bullet(\mathfrak{g}) / \mathcal{J}$$
where $\mathcal{J}$ is generated by $x \otimes y - y \otimes x - [x,y]$ for $x, y \in \mathfrak{g}$

$U^\bullet(\mathfrak{g}) = T(\mathfrak{g})/\mathcal{J}$

where $\mathcal{J}$ is generated by $x\otimes y - y\otimes x - [x,y]$ for $x,y\in\mathfrak{g}$

<u>Rmks</u>
- $S^\bullet(\mathfrak{g})$ is commutative but $U^\bullet(\mathfrak{g})$ is not.
- $S^\bullet(\mathfrak{g})$ is graded but $U^\bullet(\mathfrak{g})$ is not graded.
- However, $U^\bullet(\mathfrak{g})$ is a filtered algebra:

  Setting $T_n(\mathfrak{g}) = \bigoplus_{i\leq n} \mathfrak{g}^{\otimes i}$ defines a filtration

  $$k = T_0(\mathfrak{g}) \subset T_1(\mathfrak{g}) \subset T_2(\mathfrak{g}) \subset \cdots \subset T_n(\mathfrak{g}) \subset \cdots$$

  induces a filtration on $U^\bullet(\mathfrak{g})$

  $$k = U_0(\mathfrak{g}) \subset U_1(\mathfrak{g}) \subset U_2(\mathfrak{g}) \subset \cdots \subset U_n(\mathfrak{g}) \subset \cdots$$

  $$U_n(\mathfrak{g}) = T_n(\mathfrak{g})/\mathcal{J}$$

<u>Thm</u> (Poincaré-Birkhoff-Witt)

The symmetrization map

$$I_{PBW} : S^\bullet(\mathfrak{g}) \longrightarrow U^\bullet(\mathfrak{g})$$

$$x_1\cdots x_n \longmapsto \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} x_{\sigma(1)}\cdots x_{\sigma(n)}$$

is an isomorphism of filtered vector spaces.

<u>Rmks</u>:
- An alternative way to state the PBW Thm is as follows:

  $\longrightarrow$ I_{PBW} induces an isomorphism of (comm.) graded algebras

  $$S^\bullet(\mathfrak{g}) \xrightarrow{\cong} Gr(U^\bullet(\mathfrak{g})) = \bigoplus_{n\geq 0} U_{n+1}(\mathfrak{g})/U_n(\mathfrak{g})$$

  comm. graded algebra

- If $\{x_i\}$ is a totally ordered k-basis of $\mathfrak{g}$, and $\iota : \mathfrak{g} \longrightarrow U^\bullet(\mathfrak{g})$ is the canonical map, then the PBW Thm implies that

$$\{ \iota(x_1)\iota(x_2)\cdots\iota(x_n) : x_1 \leq x_2 \leq \cdots \leq x_n \}$$

forms a k-basis of $U^\bullet(\mathfrak{g})$.

In particular, the map $\iota : \mathfrak{g} \longrightarrow U^\bullet(\mathfrak{g})$ is injective.

Further properties

We consider actions of $\mathfrak{g}$ on $S^\bullet(\mathfrak{g})$ and $U^\bullet(\mathfrak{g})$:

- $\mathfrak{g}$ acts on $\mathfrak{g}$ by the adjoint action

  i.e. $\mathfrak{g} \curvearrowright \mathfrak{g}$ by $x \mapsto \mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$

  $$y \mapsto \mathrm{ad}_x(y) = [x,y]$$

  Extend this action to $S^\bullet(\mathfrak{g})$ by the Leibniz rule,

  namely, $\mathrm{ad}_x(y^n) = n[x,y]y^{n-1}$

  $\rightsquigarrow \mathfrak{g} \curvearrowright S^\bullet(\mathfrak{g})$

- On the other hand, $\mathfrak{g} \curvearrowright U^\bullet(\mathfrak{g})$ by $\mathrm{ad}_x(u) = xu - ux$.

Then actually I_{PBW} is equivariant w.r.t. these actions,

i.e the following diagram commutes

$$\begin{tikzcd}
S^\bullet(\mathfrak{g}) \arrow[r, "I_{PBW}"] \arrow[d, "\mathrm{ad}_x"'] \arrow[dr, phantom, "\circlearrowright"] & U^\bullet(\mathfrak{g}) \arrow[d, "\mathrm{ad}_x"] \\
S^\bullet(\mathfrak{g}) \arrow[r, "I_{PBW}"] & U^\bullet(\mathfrak{g})
\end{tikzcd}$$

Therefore, I_{PBW} restricts to an isomorphism of vect. sp.

$$I_{PBW} : S^\bullet(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\cong} U^\bullet(\mathfrak{g})^{\mathfrak{g}} = Z(U^\bullet(\mathfrak{g}))$$

Now both $S^\bullet(\mathfrak{g})^{\mathfrak{g}}$ and $U^\bullet(\mathfrak{g})^{\mathfrak{g}}$ are commutative, but

I_{PBW} is still **NOT** an algebra isomorphism.

Question : Can we correct this map to make it an algebra isom. ?

Ans : Yes, using Duflo's correction.

Duflo element

completed symm. powers

Define $J \in \hat{S}^\bullet(\mathfrak{g}^*)$ (the set of formal power series on $\mathfrak{g}$)

by $J(x) := \det\left(\frac{1-e^{-ad_x}}{ad_x}\right)$

expressed as a formal power in $c_k := tr(ad^k) \in S^k(\mathfrak{g}^*)$

$\det(e^A) = e^{tr(A)}$

$ad \in Hom(\mathfrak{g}, End(\mathfrak{g}))$
$= \mathfrak{g}^* \otimes End(\mathfrak{g})$

Rmks • $\mathfrak{g} \curvearrowright \mathfrak{g}^*$ by the coadjoint action

$\rightsquigarrow \mathfrak{g} \curvearrowright S^\bullet(\mathfrak{g}^*)$

$\Rightarrow ad^k \in (\mathfrak{g}^*)^{\otimes k} \otimes End(\mathfrak{g})$

$\Rightarrow tr(ad^k) \in (\mathfrak{g}^*)^{\otimes k}$

Fact : c_k is $\mathfrak{g}$-invariant

We regard $tr(ad^k)$ as an elt in $S^k(\mathfrak{g}^*)$ via the projection $T^\bullet(\mathfrak{g}) \to S^\bullet(\mathfrak{g})$

• The function (or formal power series)

$$\frac{1-e^{-x}}{x} = \frac{\sinh(x/2)}{x/2} = 1 - \frac{x}{2} + \frac{x^2}{6} - \frac{x^3}{24} + \dots$$

or its reciprocal

Bernoulli numbers

$$\frac{x}{1-e^{-x}} = 1 + \frac{x}{2} + \sum_{i=1}^{\infty} \frac{(-1)^{i-1} B_i}{(2i)!} x^{2i}$$

appears in the Baker-Campbell-Hausdorff (BCH) formula (~ noncommutativity in Lie groups $e^A \cdot e^B \neq e^{A+B}$)

Now $\xi \in \mathfrak{g}^* \curvearrowright S^\bullet(\mathfrak{g})$ as a derivation, namely, for $x \in \mathfrak{g}$, $\xi \cdot x^n = n\,\xi(x)\,x^{n-1}$

$\rightsquigarrow \xi^k \in S^k(\mathfrak{g}^*) \curvearrowright S^\bullet(\mathfrak{g})$ by

$$\xi^k \cdot x^n = n(n-1)\cdots(n-k+1)\,\xi(x)^k\,x^{n-k}$$

$$\longrightarrow \quad \hat{S}^\bullet(\mathfrak{g}^*) \curvearrowright S^\bullet(\mathfrak{g})$$

In particular, we have a map

$$J^{\frac{1}{2}} : S^\bullet(\mathfrak{g})^{\mathfrak{g}} \longrightarrow S^\bullet(\mathfrak{g})^{\mathfrak{g}}$$ $\leftarrow \because c_k$ is $\mathfrak{g}$-inv.

Thm (Duflo 1977)

The composition

$$I_{PBW} \circ J^{\frac{1}{2}} : S^\bullet(\mathfrak{g})^{\mathfrak{g}} \longrightarrow U^\bullet(\mathfrak{g})^{\mathfrak{g}}$$

is an isomorphism of algebras